Vertex-magic Graphs

Which graphs have a vertex-magic total labeling?

Which integers can be magic constants?

The work is very accessible and there remain many open problems!

Perhaps some of them will be solved by the end of the conference!

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\[ h = a + b + c \]
\[ h' = (7 - a) + (7 - b) + (7 - c) \]

Duality – works for regular graphs
\[4 + 5 + 6 + 2(1 + 2 + 3) \leq 3h \leq 1 + 2 + 3 + 2(4 + 5 + 6)\]
\( C_{2m} \) does not have a strong VMTL

\[
4h = (1 + 2 + 3 + 4) + 2(a + b + c + d)
= 10 + 2(26)
\]

\( h = 15.5 \quad \text{No way!} \)
For the cycle $C_n$, if $h$ is the magic constant of a VMTL, then:

$$\frac{5n + 3}{2} \leq h \leq \frac{7n + 3}{2}$$

Open Problem: Which of these feasible values are actually magic constants?
A graph that is not vertex-magic

Vertex-magic total labelings of graphs,
MacDougall, Miller, Slamin, Wallis,
A Regular graph that is not magic

If magic, then $x + y = z + y \Rightarrow x = z$ (oops)

MacDougall’s conjecture:

Every regular graph of degree at least 2, except for ??? is vertex-magic.
$K_{m,n}$ is vertex-magic iff $|m - n| \leq 1$
Using an $n \times n$ magic square to label $K_{n-1,n-1}$

\[
\begin{array}{ccc}
4 & 3 & 8 \\
9 & 5 & 1 \\
2 & 7 & 6 \\
\end{array}
\]

*Vertex-magic total labelings of graphs,*
MacDougall, Miller, Slamin, Wallis,
Using an $3 \times 3$ magic square to label $K_{2,2}$

$K_{2,2}$ is vertex-magic
Open Problem: Is there a VMTL for $K_{n,n}$ with constant $h$, for each $h$ satisfying:

$$(n+1)^3 - n^2 \leq 2h \leq (n+1)^3 + n^2$$

Regular graphs are the most interesting! (or maybe not!)


Google: Gallian dynamic survey

Two drawings of the Petersen Graph
Theorem: If $G$ is a cubic graph with a perfect matching, then $G$ is magic.

MacDougall: Every regular graph of degree at least 2 is vertex-magic, except for_____


Theorem (Gray): If G is a graph of order n with a spanning subgraph H which possesses a strong VMTL and G—E(H) is even-regular, then G also possesses a strong VMTL.

Gray’s program: To prove that all odd order regular graphs are vertex-magic, it suffices to prove that all odd order 2-regular graphs are strong vertex-magic


$C_3 \cup C_4$ does not have a strong VMTL.
$3C_3 \cup C_4$ does not have a strong VMTL.

$2C_3 \cup C_5$ does not have a strong VMTL.
Computer:

All other odd order 2-regular graphs of order up to 15 have strong VMTLs.

All triangle free odd order 2-regular graphs of order up to 17 have strong VMTLs.

Conjecture: A 2–regular graph of odd order possesses a strong VMTL if and only if it is not of the form

\[(2t - 1)C_3 \cup C_4 \quad \text{or} \quad (2t)C_3 \cup C_5\]
Resolution of Conjecture: **Oops!** (but not really!)
All graphs conjectured not to have strong VMTLs actually have them, except the 3 they found.

J. Holden, D. McQuillan, J. McQuillan,

Shifted Kotzig array:

\[
\begin{array}{cccccccc}
-3 & -2 & -1 & 0 & 1 & 2 & 3 \\
3 & 1 & -1 & -3 & 2 & 0 & -2 \\
0 & 1 & 2 & 3 & -3 & -2 & -1 \\
\end{array}
\]

Shifted Kotzig array:

\[
\begin{array}{cccccccc}
-m & -(m-1) & \cdots & 0 & 1 & 2 & \cdots & m \\
m & m-2 & \cdots & -m & m-1 & m-3 & \cdots & -(m-1) \\
0 & 1 & \cdots & m & -m & -m+1 & \cdots & -1 \\
\end{array}
\]
But can we please have a column of 0's?

1  2  3
2  -3  -1
-3  1  -2

YES! ALWAYS! This helps us to navigate through the tough technical terrain!
Kotzig completion problem:
Given a $j \times k$ array $A$ with constant column sum, and $k' \geq k$, can $A$ be completed to a $j \times k'$ Kotzig array?

Lemma (from H.M.M.M.): For each positive integer $m \geq 3$, there is a $3 \times (2m + 1)$ shifted Kotzig array such that three of the columns are:

\[
\begin{array}{ccc}
-m & 1 & m \\
0 & m-1 & -1 \\
m & -m & -(m-1)
\end{array}
\]
Vertex names

Add names: $\lambda$

"Smaller" difference: $\mu$
Subtract 1 from vertex labels

\[ \lambda^*(v) = 1 \]

\[ \lambda^*(u) = 4 \]

Switch!

D. McQuillan and K. Smith,

**Vertex-magic total labeling of odd complete graphs**, 
*Discrete Mathematics* 305 (2005), pp. 240-249.

Also:

**Vertex-magic total labeling of multiple complete graphs**, 
Wallis’ theorem:

$G$ regular, vertex-magic $\Rightarrow sG$ vertex-magic provided that degree is odd or $s$ is odd.

Is the graph $sK_n$ vertex-magic, if $n$ is odd and $s$ is even?

Only works if $n \geq 5$ used to label $K_{2n}$.
Theorem: For each odd $m \geq 5$, the set $\{0,1,\ldots,2m-1\}$ has a $(p,1)$–partition for each $p = 0,1,\ldots,2m-4$.

At least $n^2 - 4n + 1$ of the $n^2 - n$ feasible values for $2K_n$ are in the spectrum for odd $n \geq 5$. 
Conjecture: Let $p \geq 5$. Then $K_p$ has a VMTL with magic constant $h$ iff $h$ is an integer satisfying

$$(p/4)(p^2 + 3) \leq h \leq (p/4)(p+1)^2$$

For $p \geq 6$, and $p \equiv 2 \mod 4$, we got these ones:

$$(p/4)(p^2 + 6) \leq h \leq (p/4)(p^2 + 2p - 2)$$

**Conjecture**: There is an integer $M$ such that each regular graph of order at least $M$ has the property that it possesses VMTLs with each of its feasible values for magic constants.

A. Armstrong and D. McQuillan,
$2K_3$ is not vertex-magic.

Part of a VMTL of $14K_3$ with $h = 146$. 
D. McQuillan and J. McQuillan, *Magic labelings of triangles*, 

At least $s + 1$ of the $3s + 1$ feasible values are in the spectrum for odd $s$.

At least $2s - 2$ of the $3s$ feasible values are in the spectrum for even $s \geq 6$

If $s = 2 \cdot 3^k$, $k \geq 1$, then we win!

$$(1/2)(15s + 4) \leq h \leq (1/2)(21s + 2)$$
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