On the value of making up your own question

Dan McQuillan

First, we consider an example of how one might make up a question.

Here is a nice fact:

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$$

Let's have some fun with this fact by making it the answer of a question. Think of it as a mystery equation, and it's our job to point the problem-solver in the right direction to find it. Hmm…

The sum of these three denominators is 11. After a bit of thought we see that $\{2,3,6\}$ is the only set of 3 positive integers with sum 11, such that the sum of reciprocals is equal to 1. So we will tell the problem-solver that we have x numbers whose sum is $f(x)$ and we'll pick a function $f(x)$ such that

$$f(3)=11.$$

We'll also mention that the sum of reciprocals is exactly 1.

Now we need a hint so that the solver can be directed towards $x=3$.

Consider the more general situation where the x numbers are only required to be positive real numbers instead of positive integers. If the sum of reciprocals is 1, then it is well known that the sum, $f(x)$, is minimized if the x numbers are equal to each other, and thus equal to x (think of famous inequalities!). Hence

$$x+x+\cdots+x=x^2 \le f(x)$$

Let's pick

$$f(x)=5x-4$$

Therefore,

$$x^2 \le 5x-4$$

Solving this inequality yields:

$$1 \le x \le 4$$

We are ready to state the problem. Here's a version of it, close to the way it appeared on the *2005 William Lowell Putnam Mathematical Competition*:

Problem: Find all positive real integers $x, k_1, k_2, ..., k_x$ such that $k_1 + k_2 + \cdots + k_x = 5x - 4$ and such that

$$\frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_x} = 1.$$

The reader now knows how to solve this. Our mystery equation comes from an analysis of the case where $x = 3$. There are a couple of other cases to consider, corresponding to $x = 1$ and $x = 4$.

To be clear, I did not make up this question. I also have no knowledge of what was going through the mind of the person who did make up this question. I was merely having fun with this beautiful problem!

My point is that this question, while incredibly difficult, may not seem so bad from the point of view of the *problem-maker*. It's only difficult for the problem-solver. More importantly, I believe that the problem-solver has a much better chance of answering this type of question *after* having experience as a problem-maker. As much as possible, we want to be proactive rather than reactive.

So I propose that we practice making up good questions. Good luck on the Putnam!

Originally posted November 30, 2012 on Danmcquillan.com

(A version of this post will be included in an up-coming book I'm working on, about mathematical problem-solving).

Epilogue December 2nd, 2012: We were thrilled to have our best ever attendance at the Putnam yesterday! Nine enthusiastic students participated in *many* hours of mathematical exploration!